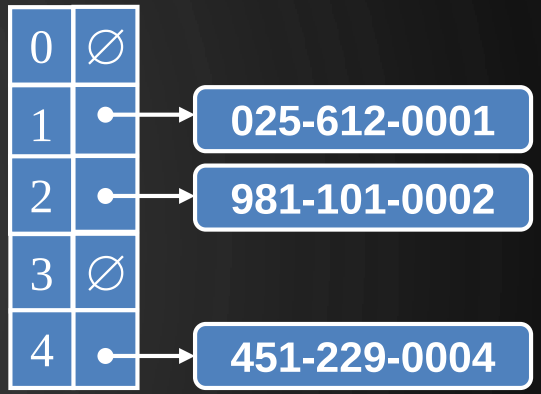


CHAPTER 9
HASH TABLES, MAPS, AND SKIP LISTS

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN C++, GOODRICH, TAMASSIA AND MOUNT (WILEY 2004) AND SLIDES FROM NANCY M. AMATO

READING

- Map ADT (Ch. 9.1)
- Dictionary ADT (Ch. 9.5)
- Ordered Maps (Ch. 9.3)
- Hash Tables (Ch. 9.2)

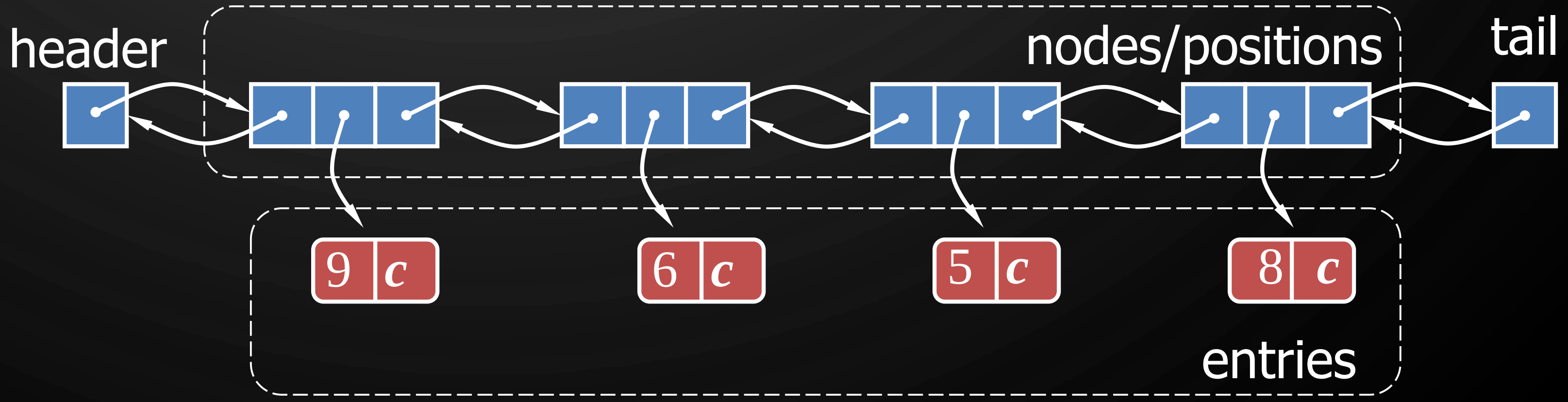
MAP ADT

- A map models a searchable collection of key-value pair (called entries)
- Multiple items with the same key are not allowed
- Applications:
  - address book or student records
  - mapping host names (e.g., cs16.net) to internet addresses (e.g., 128.148.34.101)
- Often called "associative" containers

- Map ADT methods:
  - $\text{find}(k)$ – if M has an entry $e = (k, v)$, return an iterator p referring to this entry, else, return special end iterator.
  - $\text{put}(k, v)$ – if M has no entry with key k, then add entry (k, v) to M, otherwise replace the value of the entry with v; return iterator to the inserted/modified entry
  - $\text{erase}(k)$, $\text{erase}(p)$ – remove from M entry with key k or iterator p; An error occurs if there is no such element.
  - $\text{size}()$, $\text{empty}()$, $\text{begin}()$, $\text{end}()$

LIST-BASED MAP IMPLEMENTATION

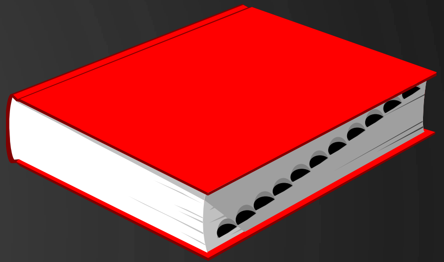
- We can imagine implementing the map with an unordered list
- $\mathrm{find}(k)$ – search the list of entries for key k
- $\mathrm{put}(k, v)$ – search the list for an existing entry, otherwise call $\mathrm{insertBack}\big((k, v)\big)$
- Similar idea for erase functions
- Complexities?
  - $O(n)$ on all



DIRECT ADDRESS TABLE MAP IMPLEMENTATION

- A direct address table is a map in which
  - The keys are in the range $[0, N]$
  - Stored in an array T of size N
  - Entry with key k stored in $T[k]$
- Performance:
  - $\text{put}(k, v)$, $\text{find}(k)$, and $\text{erase}(k)$ all take $O(1)$ time
  - Space - requires space $O(N)$, independent of n, the number of entries stored in the map
- The direct address table is not space efficient unless the range of the keys is close to the number of elements to be stored in the map, i.e., unless n is close to N.

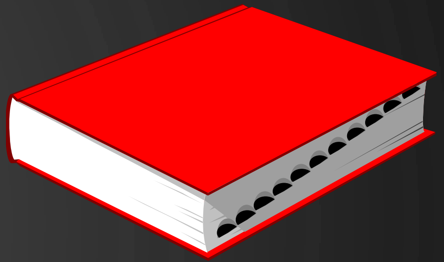
DICTIONARY ADT



- The dictionary ADT models a searchable collection of key-value entries
- The main difference from a map is that multiple items with the same key are allowed
- Any data structure that supports a dictionary also supports a map
- Applications:
  - Dictionary which has multiple definitions for the same word

- Dictionary ADT adds the following to the Map ADT:
  - $\mathrm{findAll}(k)$ – Return iterators (b, e) s.t. that all entries with key k are between them, not including e
  - $insert(k, v)$ – Insert an entry with key k and value v, returning an iterator to the newly created entry
  - Note – $\mathrm{find}(k)$, $\mathrm{erase}(k)$ operate on arbitrary entries with key k
  - Note – "$\mathrm{put}(k, v)$" doesn't exist

ORDERED MAP/DICTIONARY ADT

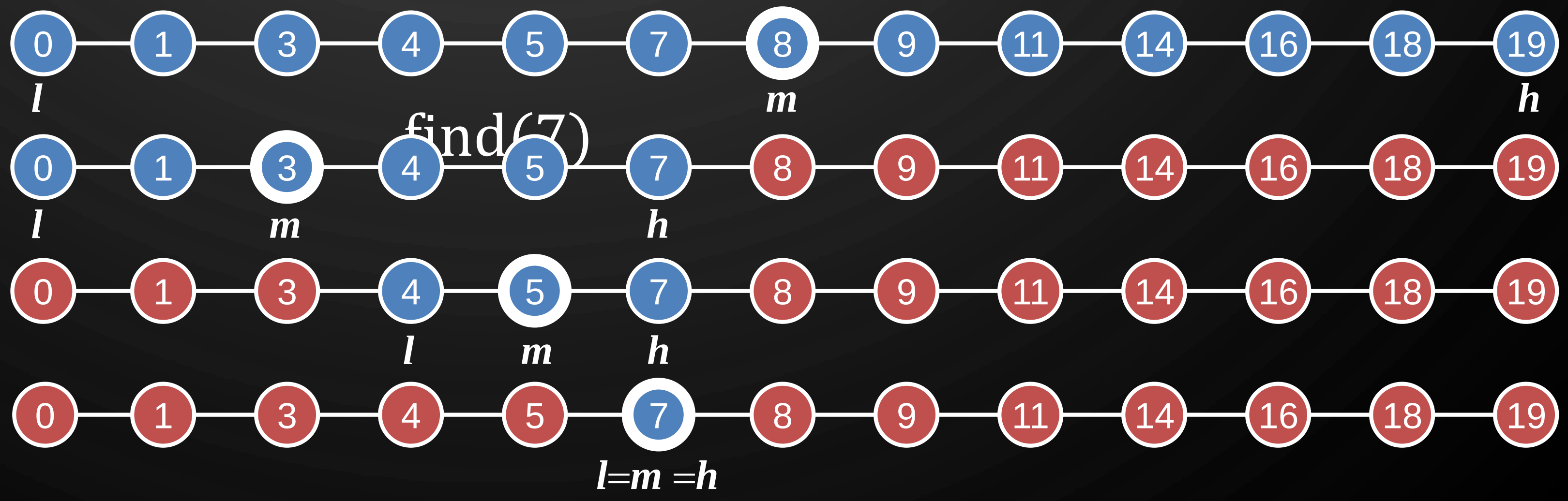


- An Ordered Map/Dictionary supports the usual map/dictionary operations, but also maintains an order relation for the keys.
- Naturally supports
  - Ordered search tables - store dictionary in a vector by non-decreasing order of the keys
  - Utilizes binary search

- Ordered Map/Dictionary ADT adds the following functionality to a map/dictionary
  - firstEntry(), lastEntry() – return iterators to entries with the smallest and largest keys, respectively
  - ceilingEntry(k), floorEntry(k) – return an iterator to the least/greatest key value greater than/less than or equal to k
  - lowerEntry(k), higherEntry(k) – return an iterator to the greatest/least key value less than/greater than k

EXAMPLE OF ORDERED MAP: BINARY SEARCH

- Binary search performs operation $\text{find}(k)$ on an ordered search table
  - similar to the high-low game
  - at each step, the number of candidate items is halved
  - terminates after a logarithmic number of steps
- Example



MAP/DICTIONARY IMPLEMENTATIONS

	$\mathbf{put}(k, v)$	$\mathbf{find}(k)$	**Space**
Unsorted list	$O(n)$	$O(n)$	$O(n)$
Direct Address Table (map only)	$O(1)$	$O(1)$	$O(N)$
Ordered Search Table (ordered map/dictionary)	$O(n)$	$O(\log n)$	$O(n)$

CH. 9.2
HASH TABLES

HASH TABLES

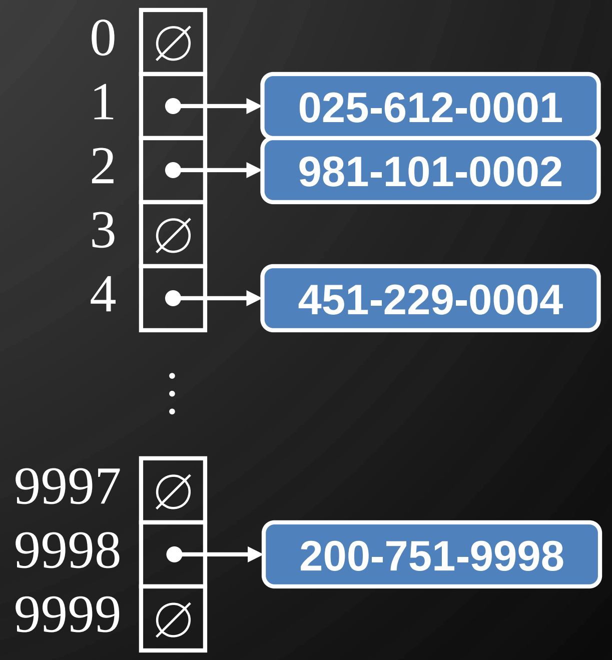
- Sometimes a key can be interpreted or transformed into an address. In this case, we can use an implementation called a hash table for the Map ADT.
- Hash tables
  - Essentially an array A of size N (either to an element itself or to a "bucket")
  - A Hash function $h(k) \rightarrow [0, N-1]$, $h(k)$ is referred to as the hash value
    - Example - $h(k) = k \bmod N$
  - Goal is to store elements (k, v) at index $i = h(k)$

ISSUES WITH HASH TABLES

- Issues
  - Collisions - some keys will map to the same index of H (otherwise we have a Direct Address Table).
    - Chaining - put values that hash to same location in a linked list (or a "bucket")
    - Open addressing - if a collision occurs, have a method to select another location in the table.
  - Load factor
  - Rehashing

EXAMPLE

- We design a hash table for a Map storing items (SSN, Name), where SSN (social security number) is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function $h(k) = \text{last four digits of } k$



HASH FUNCTIONS

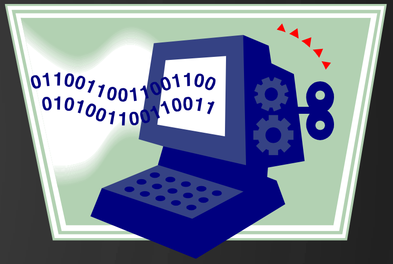
- A hash function is usually specified as the composition of two functions:
- Hash code:
  h_1: keys $\rightarrow$ integers
- Compression function:
  h_2: integers $\rightarrow [0, N-1]$

- The hash code is applied first, and the compression function is applied next on the result, i.e.,

$$h(k) = h_2\big(h_1(k)\big)$$

- The goal of the hash function is to "disperse" the keys in an apparently random way

HASH CODES



- Memory address:
  - We reinterpret the memory address of the key object as an integer
  - Good in general, except for numeric and string keys
- Integer cast:
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer type (e.g., byte, short, int and float in C++)
- Component sum:
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type (e.g., long and double in C++)

HASH CODES

- Polynomial accumulation:
  - We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)

    $$a_0 a_1 \ldots a_{n-1}$$

  - We evaluate the polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$ at a fixed value z, ignoring overflows
  - Especially suitable for strings (e.g., the choice z = 33 gives at most 6 collisions on a set of 50,000 English words)

- Cyclic Shift:
  - Like polynomial accumulation except use bit shifts instead of multiplications and bitwise or instead of addition
- Can be used on floating point numbers as well by converting the number to an array of characters

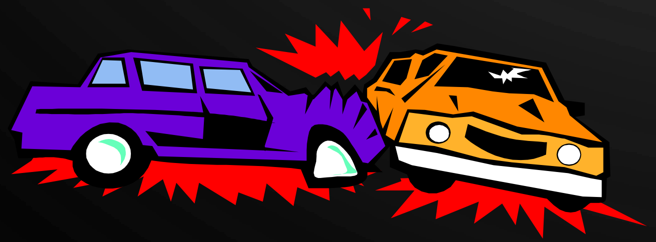
COMPRESSION FUNCTIONS

- Division:
  - $h_2(k) = |k| \bmod N$
  - The size N of the hash table is usually chosen to be a prime (based on number theory principles and modular arithmetic)
- Multiply, Add and Divide (MAD):
  - $h_2(k) = |ak + b| \bmod N$
  - a and b are nonnegative integers such that

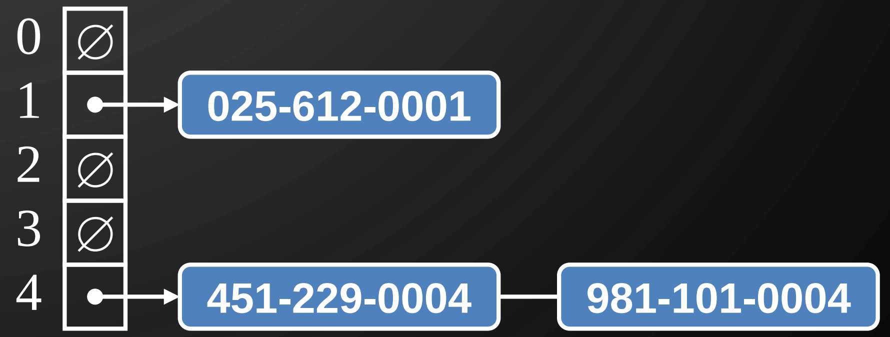
    $a \bmod N \neq 0$
  - Otherwise, every integer would map to the same value b

COLLISION RESOLUTION WITH
SEPARATE CHAINING

- Collisions occur when different elements are mapped to the same cell
- Separate Chaining: let each cell in the table point to a linked list of entries that map there



- Chaining is simple, but requires additional memory outside the table



EXERCISE
SEPARATE CHAINING

- Assume you have a hash table H with $N = 9$ slots ($A[0 - 8]$) and let the hash function be $h(k) = k \bmod N$
- Demonstrate (by picture) the insertion of the following keys into a hash table with collisions resolved by chaining
  - 5, 28, 19, 15, 20, 33, 12, 17, 10

COLLISION RESOLUTION WITH
OPEN ADDRESSING - LINEAR PROBING

- In Open addressing the colliding item is placed in a different cell of the table
- Linear probing handles collisions by placing the colliding item in the next (circularly) available table cell. So the ith cell checked is:

$$h(k,i) = |h(k) + i| \bmod N$$

- Each table cell inspected is referred to as a "probe"
- Colliding items lump together, causing future collisions to cause a longer probe sequence

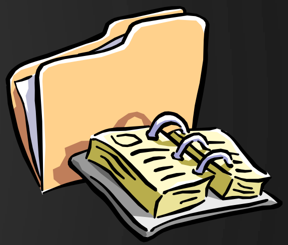
- Example:
  - $h(k) = k \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

SEARCH WITH LINEAR PROBING



- Consider a hash table A that uses linear probing
- find(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - An empty cell is found, or
    - N cells have been unsuccessfully probed

Algorithm find(k)

```
i ← h(k)
p ← 0
repeat
    c ← A[i]
    if c ≠ ∅
        return null
    else if c.key() = k
        return c
    else
        i ← (i + 1) mod N
        p ← p + 1
until p = N
return null
```

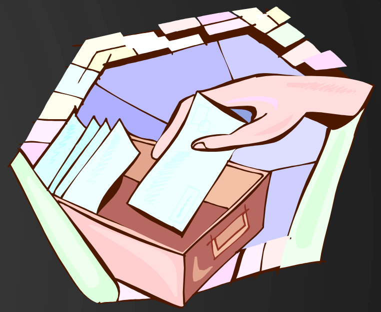
UPDATES WITH LINEAR PROBING

- To handle insertions and deletions, we introduce a special object, called AVAILABLE, which replaces deleted elements
- $\mathrm{erase}(k)$
  - We search for an item with key k
  - If such an item (k, v) is found, we replace it with the special item AVAILABLE
- $\mathrm{put}(k, v)$
  - We start at cell $h(k)$
  - We probe consecutive cells until one of the following occurs
    - A cell i is found that is either empty or stores AVAILABLE, or
    - N cells have been unsuccessfully probed

EXERCISE
OPEN ADDRESSING – LINEAR PROBING

- Assume you have a hash table H with $N = 11$ slots ($A[0 - 10]$) and let the hash function be $h(k) = k \bmod N$
- Demonstrate (by picture) the insertion of the following keys into a hash table with collisions resolved by linear probing.
  - 10, 22, 31, 4, 15, 28, 17, 88, 59

COLLISION RESOLUTION WITH
OPEN ADDRESSING – QUADRATIC PROBING



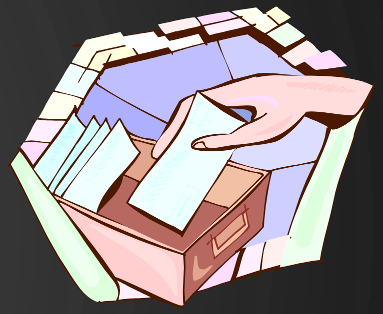
- Linear probing has an issue with clustering
- Another strategy called quadratic probing uses a hash function

$$h(k, i) = (h(k) + i^2) \bmod N$$

  for $i = 0, 1, \ldots, N-1$

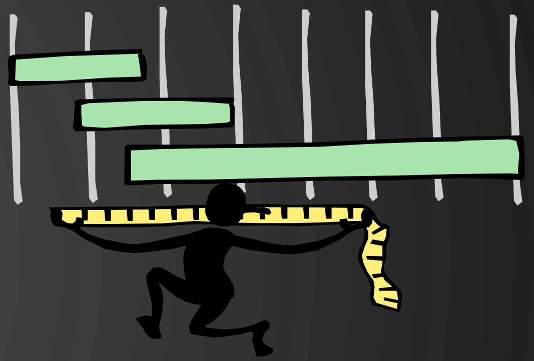
- This can still cause secondary clustering

COLLISION RESOLUTION WITH
OPEN ADDRESSING - DOUBLE HASHING



- Double hashing uses a secondary hash function $h_2(k)$ and handles collisions by placing an item in the first available cell of the series
  $$h(k,i) = \big(h_1(k) + i h_2(k)\big) \bmod N$$
  for $i = 0, 1, \ldots, N-1$
- The secondary hash function $h_2(k)$ cannot have zero values
- The table size N must be a prime to allow probing of all the cells

- Common choice of compression map for the secondary hash function:
  $$h_2(k) = q - (k \bmod q)$$
  where
  - $q < N$
  - q is a prime
- The possible values for $h_2(k)$ are $1, 2, \ldots, q$



PERFORMANCE OF HASHING

- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time
- The worst case occurs when all the keys inserted into the map collide
- The load factor $\lambda = \frac{n}{N}$ affects the performance of a hash table
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is

$$\frac{1}{1-\lambda} = \frac{1}{1-{}^{n}/_{N}} = \frac{1}{N-{}^{n}/_{N}} = \frac{N}{N-n}$$

- The expected running time of all the Map ADT operations in a hash table is $O(1)$
- In practice, hashing is very fast provided the load factor is not close to 100%
- Applications of hash tables
  - Small databases
  - Compilers
  - Browser caches

UNIFORM HASHING ASSUMPTION

- The probe sequence of a key k is the sequence of slots probed when looking for k
  - In open addressing, the probe sequence is $h(k,0), h(k,1), \ldots, h(k,N-1)$
- Uniform Hashing Assumption
  - Each key is equally likely to have any one of the $N!$ permutations of $\{0, 1, \ldots, N-1\}$ as is probe sequence
  - Note: Linear probing and double hashing are far from achieving Uniform Hashing
    - Linear probing: N distinct probe sequences
    - Double Hashing: N^2 distinct probe sequences

PERFORMANCE OF UNIFORM HASHING

- Theorem: Assuming uniform hashing and an open-address hash table with load factor $\lambda = \frac{n}{N} < 1$, the expected number of probes in an unsuccessful search is at most $\frac{1}{1-\lambda}$.
- Exercise: compute the expected number of probes in an unsuccessful search in an open address hash table with $\lambda = \frac{1}{2}$, $\lambda = \frac{3}{4}$, and $\lambda = \frac{99}{100}$.

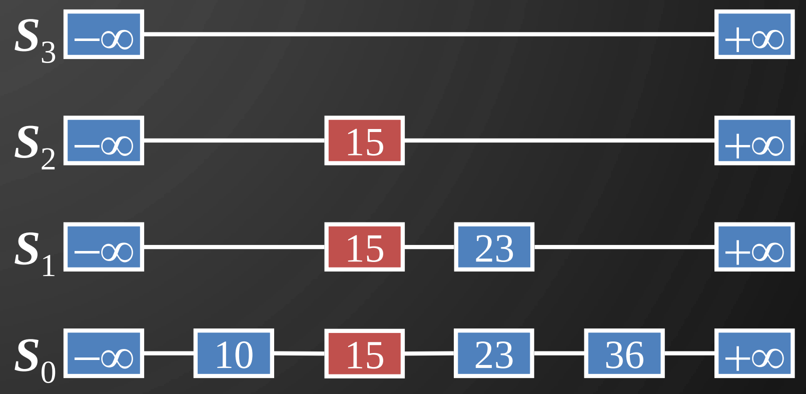
ON REHASHING

- Keeping the load factor low is vital for performance
- When resizing the table:
  - Reallocate space for the array
  - Design a new hash function (new parameters) for the new array size
  - For each item you reinsert it into the table

SUMMARY MAPS/DICTIONARIES (SO FAR)

	$\text{put}(k, v)$	$\text{find}(k)$	Space
Log File	$O(1)$	$O(n)$	$O(n)$
Direct Address Table (map only)	$O(1)$	$O(1)$	$O(N)$
Lookup Table (ordered map/dictionary)	$O(n)$	$O(\log n)$	$O(n)$
Hashing (chaining)	$O(1)$	$O(n/N)$	$O(n+N)$
Hashing (open addressing)	$O\left(\frac{1}{1-\frac{n}{N}}\right)$	$O\left(\frac{1}{1-\frac{n}{N}}\right)$	$O(N)$

CH. 9.4
SKIP LISTS



RANDOMIZED ALGORITHMS

- A randomized algorithm controls its execution through random selection (e.g., coin tosses)
- It contains statements like:

  $b \leftarrow \text{randomBit}()$

  if $b = 0$

  do something…

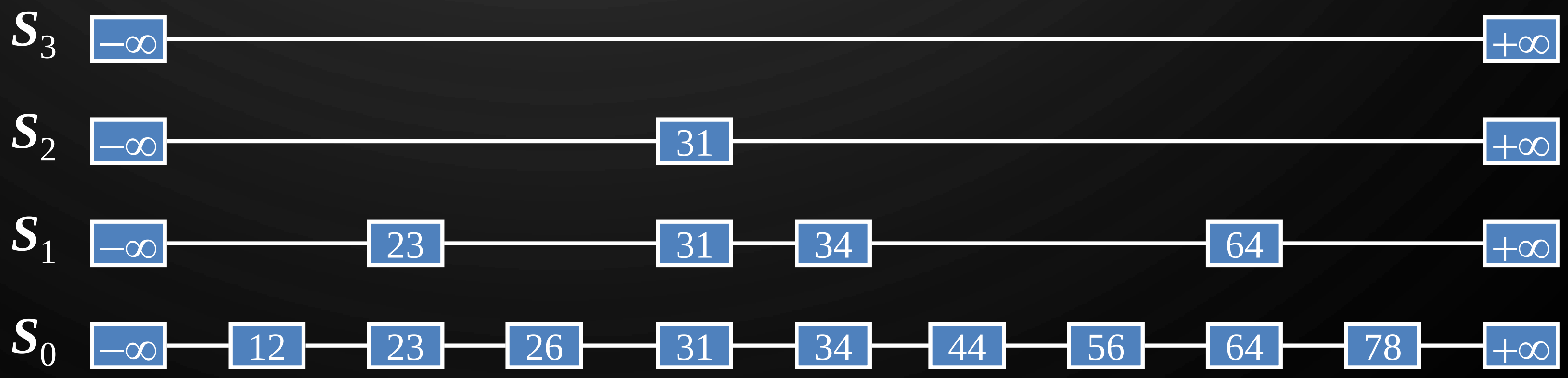
  else $//b = 1$

  do something else…
- Its running time depends on the outcomes of the "coin tosses"

- Through probabilistic analysis we can derive the expected running time of a randomized algorithm
- We make the following assumptions in the analysis:
  - the coins are unbiased
  - the coin tosses are independent
- The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give "heads")
- We use a randomized algorithm to insert items into a skip list to insert in expected $O(\log n)$–time
- When randomization is used in data structures they are referred to as probabilistic data structures

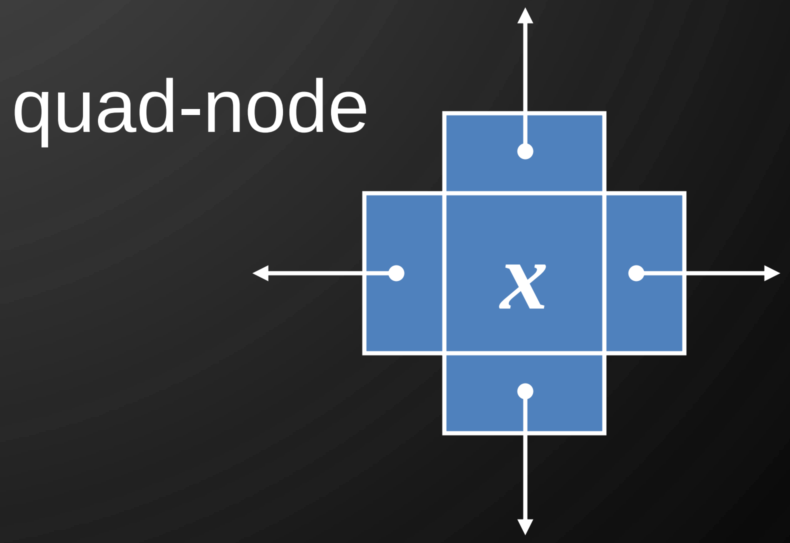
WHAT IS A SKIP LIST?

- A skip list for a set S of distinct (key, element) items is a series of lists
$$S_0, S_1, \dots, S_h$$
  - Each list S_i contains the special keys $+\infty$ and $-\infty$
  - List S_0 contains the keys of S in non-decreasing order
  - Each list is a subsequence of the previous one, i.e.,
$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$$
  - List S_h contains only the two special keys
- Skip lists are one way to implement the Ordered Map ADT
- Java applet



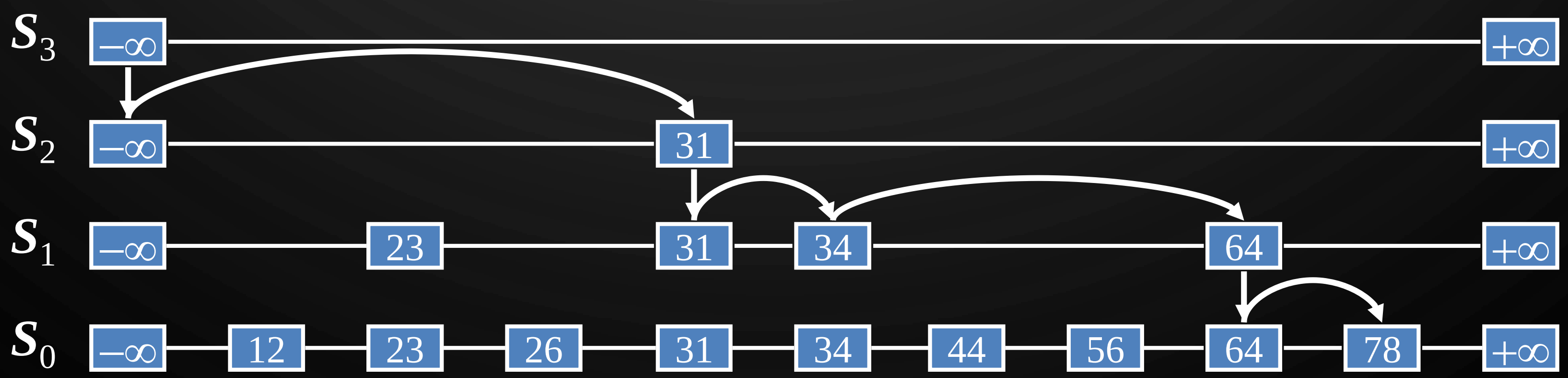
IMPLEMENTATION

- We can implement a skip list with quad-nodes
- A quad-node stores:
  - (Key, Value)
  - links to the nodes before, after, below, and above
- Also, we define special keys $+\infty$ and $-\infty$, and we modify the key comparator to handle them



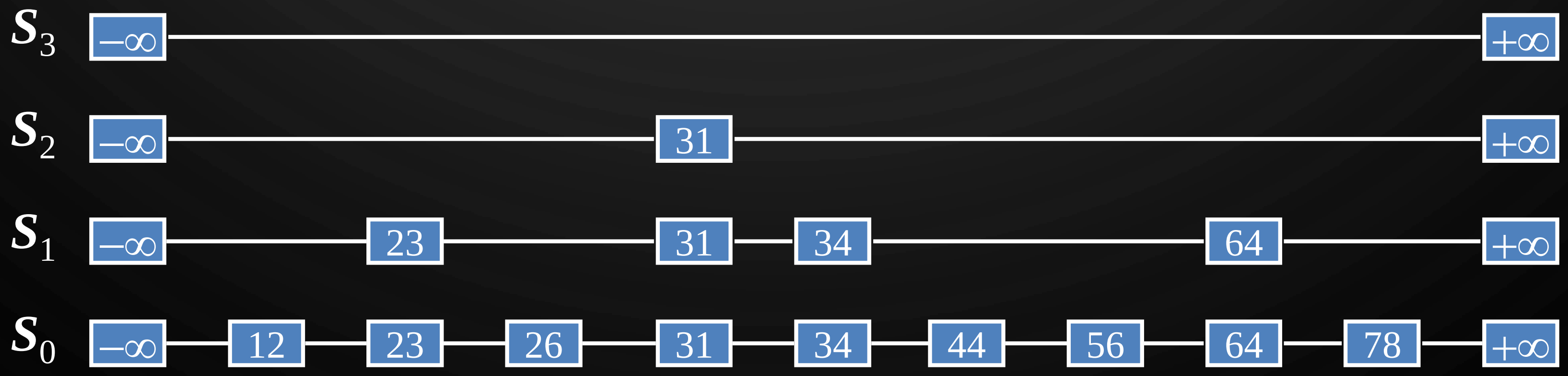
SEARCH - FIND(k)

- We search for a key k in a skip list as follows:
  - We start at the first position of the top list
  - At the current position p, we compare k with $y \leftarrow p.\text{next}().\text{key}()$
    $x = y$: we return $p.\text{next}().\text{value}()$
    $x > y$: we scan forward
    $x < y$: we drop down
  - If we try to drop down past the bottom list, we return ***NO_SUCH_KEY***
- Example: search for 78



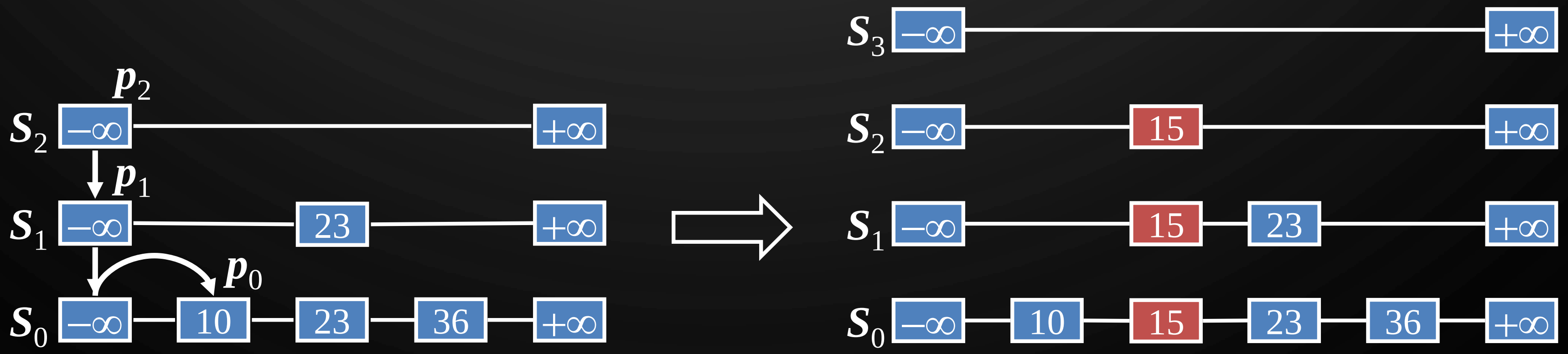
EXERCISE
SEARCH

- We search for a key k in a skip list as follows:
  - We start at the first position of the top list
  - At the current position p, we compare k with $y \leftarrow p.\text{next}().\text{key}()$
    $x = y$: we return $p.\text{next}().\text{value}()$
    $x > y$: we scan forward
    $x < y$: we drop down
  - If we try to drop down past the bottom list, we return ***NO_SUCH_KEY***
- Ex 1: search for 64: list the (S_i, node) pairs visited and the return value
- Ex 2: search for 27: list the (S_i, node) pairs visited and the return value



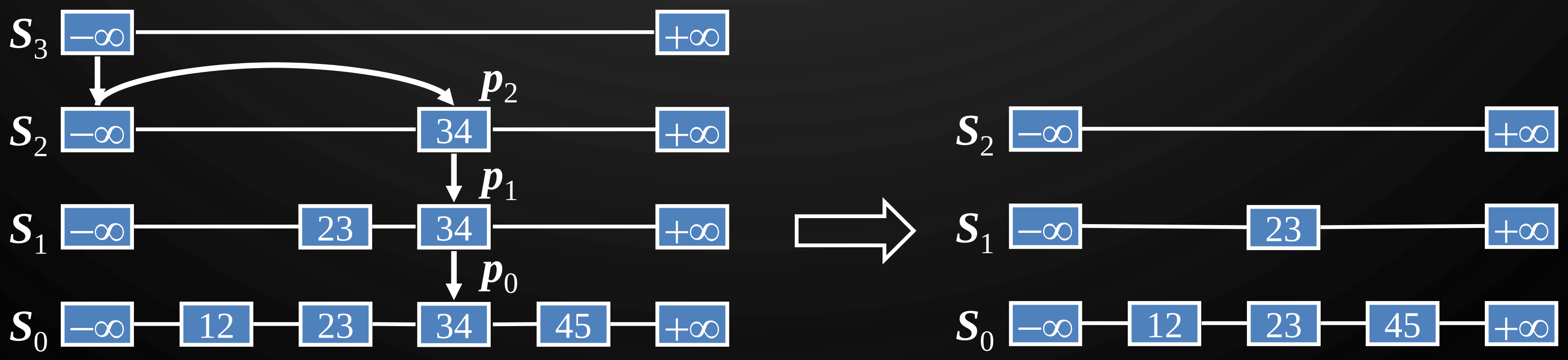
INSERTION - $\text{PUT}(k, v)$

- To insert an item (k, v) into a skip list, we use a randomized algorithm:
  - We repeatedly toss a coin until we get tails, and we denote with i the number of times the coin came up heads
  - If $i \geq h$, we add to the skip list new lists $S_{h+1}, \dots, S_{i+1}$ each containing only the two special keys
  - We search for k in the skip list and find the positions $p_0, p_1, \dots, p_i$ of the items with largest key less than k in each list $S_0, S_1, \dots, S_i$
  - For $i \leftarrow 0, \dots, i$, we insert item (k, v) into list S_i after position p_i
- Example: insert key 15, with $i = 2$



DELETION - ERASE(k)

- To remove an item with key k from a skip list, we proceed as follows:
  - We search for k in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with key k, where position p_i is in list S_i
  - We remove positions $p_0, p_1, \ldots, p_i$ from the lists $S_0, S_1, \ldots, S_i$
  - We remove all but one list containing only the two special keys
- Example: remove key 34



SPACE USAGE

- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
- We use the following two basic probabilistic facts:
  - Fact 1: The probability of getting i consecutive heads when flipping a coin is $\frac{1}{2^i}$
  - Fact 2: If each of n items is present in a set with probability p, the expected size of the set is np

- Consider a skip list with n items
  - By Fact 1, we insert an item in list S_i with probability $\frac{1}{2^i}$
  - By Fact 2, the expected size of list S_i is $\frac{n}{2^i}$
- The expected number of nodes used by the skip list is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n$$

- Thus the expected space is $O(2n)$

HEIGHT

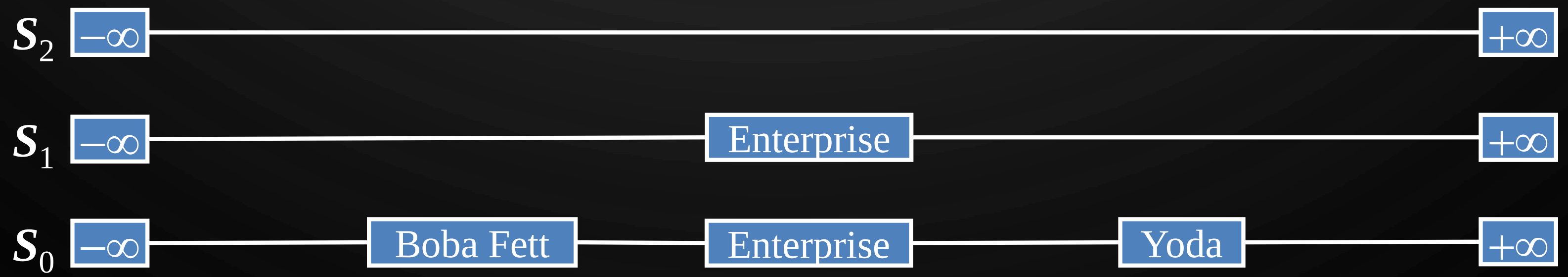
- The running time of $\mathrm{find}(k)$, $\mathrm{put}(k, v)$, and $\mathrm{erase}(k)$ operations are affected by the height h of the skip list
- We show that with high probability, a skip list with n items has height $O(\log n)$
- We use the following additional probabilistic fact:
  - Fact 3: If each of n events has probability p, the probability that at least one event occurs is at most np
- Consider a skip list with n items
  - By Fact 1, we insert an item in list S_i with probability $\frac{1}{2^i}$
  - By Fact 3, the probability that list S_i has at least one item is at most $\frac{n}{2^i}$
- By picking $i = 3 \log n$, we have that the probability that $S_{3 \log n}$ has at least one item is
  at most $\frac{n}{2^{3 \log n}} = \frac{n}{n^3} = \frac{1}{n^2}$
- Thus a skip list with n items has height at most $3 \log n$ with probability at least $1 - \frac{1}{n^2}$

SEARCH AND UPDATE TIMES

- The search time in a skip list is proportional to
  - the number of drop-down steps
  - the number of scan-forward steps
- The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ expected time
- To analyze the scan-forward steps, we use yet another probabilistic fact:
  - Fact 4: The expected number of coin tosses required in order to get tails is 2
- When we scan forward in a list, the destination key does not belong to a higher list
  - A scan-forward step is associated with a former coin toss that gave tails
- By Fact 4, in each list the expected number of scan-forward steps is 2
- Thus, the expected number of scan-forward steps is $O(\log n)$
- We conclude that a search in a skip list takes $O(\log n)$ expected time
- The analysis of insertion and deletion gives similar results

EXERCISE

- You are working for ObscureDictionaries.com a new online start-up which specializes in sci-fi languages. The CEO wants your team to describe a data structure which will efficiently allow for searching, inserting, and deleting new entries. You believe a skip list is a good idea, but need to convince the CEO. Perform the following:
  - Illustrate insertion of “X-wing” into this skip list. Randomly generated (1, 1, 1, 0).
  - Illustrate deletion of an incorrect entry “Enterprise”
  - Argue the complexity of deleting from a skip list



SUMMARY

- A skip list is a data structure for dictionaries that uses a randomized insertion algorithm
- In a skip list with n items
  - The expected space used is $O(n)$
  - The expected search, insertion and deletion time is $O(\log n)$
- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability
- Skip lists are fast and simple to implement in practice